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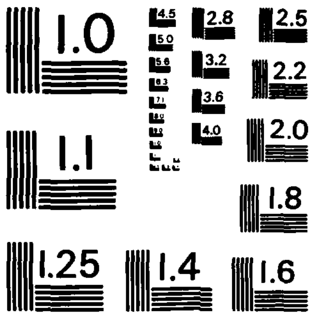


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# A POWER TRANSFORMATION EXPONENTIAL REGRESSION MODEL FOR CENSORED FAILURE TIME DATA

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# A POWER TRANSFORMATION EXPONENTIAL REGRESSION MODEL FOR CENSORED FAILURE TIME DATA




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## ABSTRACT


A power transformation regression model is considered for exponentailly distributed time to failure data with right censoring, procedures for estimation of parameters by maximum likelihood and assissment of goodness of model fit are described and illustrated.



Key Words: Power transformation model, censored survival data, exponential distribution.

## 1. INTRODUCTION

Suppose that we have $n$ individuals or components which are subject to 'failure' and that the response variable of interest is time to failure. Let $T_1^*, T_2^*, \ldots, T_n^*$ be independent random variables representing the times to failure of the individuals. We suppose that right censoring may occur because of the need for early termination of the investigation and let $T_1, T_2, \ldots, T_n$ represent the recorded survival times. Defining censoring indicator variables

$$\omega_i = \begin{cases} 1 & \text{if } T_i^* \text{ is uncensored} \\ 0 & \text{if } T_i^* \text{ is censored} \end{cases} \tag{1.1}$$

we have

$$T_i^* = T_i \text{ if } \omega_i = 1\,, \quad T_i^* > T_i \text{ if } \omega_i = 0 \tag{1.2}$$

We let

$$n_u = \sum_{i=1}^{n} \omega_i \quad , \qquad n_c = \sum_{i=1}^{n} (1 - \omega_i) \tag{1.3}$$

denote the numbers of uncensored and censored observations, respectively.

We now suppose that measurements are available on $k$ explanatory variables $x_1, x_2, \ldots, x_k$. Setting $\underset{\sim}{x}' = (x_1, x_2, \ldots, x_k)$ the p.d.f. and survival function of $T^*$ given $\underset{\sim}{x}$ will be denoted by $f(t;\underset{\sim}{x})$ and $S(t;\underset{\sim}{x})$, respectively. If the hazard function $f(t;\underset{\sim}{x})/S(t;\underset{\sim}{x})$ is assumed to be a constant independent of $t$ for any given $\underset{\sim}{x}$, then $T^*$ has the exponential distribution with p.d.f.

$$f(t;\underset{\sim}{x}) = \begin{cases} \mu_{\underset{\sim}{x}}^{-1} \exp(-t/\mu_{\underset{\sim}{x}}), & t > 0 \\ 0 \text{ otherwise} \end{cases} \tag{1.4}$$

where $\mu_{\underset{\sim}{x}}$ denotes the expected value of $T^*$ given $\underset{\sim}{x}$.

Various models have been proposed in the literature to represent the dependence of $\mu_{\underset{\sim}{x}}$ on $\underset{\sim}{x}$. Fiegl and Zelen (1965) consider the model form

$$\mu_{\underset{\sim}{x}} = \lambda(1 + \underset{\sim}{x}'\underset{\sim}{\beta}) \tag{1.5}$$

while Greenberg et. al. (1974) use the form

$$\mu_{\underset{\sim}{x}} = \lambda/(1 + \underset{\sim}{x}'\underset{\sim}{\beta}) \tag{1.6}$$

where $\underset{\sim}{\beta}' = (\beta_1, \beta_2, \ldots, \beta_k)$ and $\lambda$ is a positive constant. Both models require that the condition $\underset{\sim}{x}'\underset{\sim}{\beta} < 1$ must be imposed to ensure that $\mu_{\underset{\sim}{x}} > 0$. An alternative model which does not require a constraint to be imposed on $\underset{\sim}{x}'\underset{\sim}{\beta}$ is

$$\mu_{\underset{\sim}{x}} = \lambda \exp(\underset{\sim}{x}'\underset{\sim}{\beta}) \tag{1.7}$$

This model arises for the exponential case from the well-known family of proportional hazard regression models (see e.g. Kay (1977)) in which an assumed underlying hazard function is adjusted by multiplicative exponential factors to allow for the effect of the explanatory variables. Prentice (1973) also discusses the use of censored regression models for the exponential case.

In this paper, we consider the power transformation model given by

$$\mu_{\underset{\sim}{x}} = \lambda(1 + \delta\underset{\sim}{x}'\underset{\sim}{\beta})^{1/\delta} \tag{1.8}$$

We shall refer to $\delta$ as the power parameter. It is seen that when $\delta = 1$, the model corresponds to the usual additive regression model for the mean given by (1.5), while if $\delta = -1$ the reciprocal model (1.6) is obtained after appropriate reparameterisation. When $\delta \to 0$, the model given by (1.7) is obtained. For later work, it is convenient to write the power transformation model in the form

$$\mu_{\underset{\sim}{x}} = (1 + \delta\underset{\sim}{x}_*'\underset{\sim}{\beta}_*)^{1/\delta} \tag{1.9}$$

where $\underset{\sim}{x}_*' = (1, \underset{\sim}{x}')$, $\underset{\sim}{\beta}_*' = (\beta_{*0}, \beta_{*1}, \ldots, \beta_{*k})$ and

$$\beta_{*0} = (\lambda^{\delta} - 1)/\delta, \qquad \beta_{*j} = \lambda^{\delta}\beta_j, \; j = 1, \ldots, k \tag{1.10}$$

In general, the power parameter $\delta$ as well as the coefficient vector $\underset{\sim}{\beta}_*$ will have to be estimated from the data. In section 2, results are given for maximum likelihood estimation and it is shown how the estimates can be obtained numerically using the statistical package GLIM. The problem of assessment of the goodness of fit of the model is discussed briefly in section 3 and in section 4 a numerical example is used to illustrate the procedure.

## 2. MAXIMUM LIKELIHOOD ESTIMATION

We shall assume that the stochastic model underlying the censoring mechanism is unknown. The likelihood given the data $(t_i, \omega_i, \underset{\sim}{x}_i)$, where $\underset{\sim}{x}_i' = (x_{i1}, x_{i2}, \ldots, x_{ik})$, is

$$L = \prod_{i=1}^{n} \left\{ f(t_i ; \underset{\sim}{x}_i) \right\}^{\omega_i} \left\{ S(t_i ; \underset{\sim}{x}_i) \right\}^{1-\omega_i}$$

$$= \prod_{i=1}^{n} \left\{ \mu_{\underset{\sim}{x}_i}^{-\omega_i} \exp(-t_i/\mu_{\underset{\sim}{x}_i}) \right\} \qquad (2.1)$$

where $\mu_{\underset{\sim}{x}_i} = (1 + \delta \underset{\sim}{x}_{*i}' \underset{\sim}{\beta}_*)^{1/\delta}$ and $\underset{\sim}{x}_{*i}' = (1, \underset{\sim}{x}_i')$. If we put

$$\mu_i = t_i \mu_{\underset{\sim}{x}_i}^{-1} , \qquad i=1,2,\ldots,n \qquad (2.2)$$

the log likelihood is

$$\log L = \sum_{i=1}^{n} (\omega_i \log \mu_i - \mu_i) - \sum_{i=1}^{n} \omega_i \log t_i \qquad (2.3)$$

The likelihood equations are obtained by setting

$$\sum_{i=1}^{n} \frac{(\omega_i - \mu_i)}{\mu_i} \frac{\partial \mu_i}{\partial \delta} = 0 , \quad \sum_{i=1}^{n} \frac{(\omega_i - \mu_i)}{\mu_i} \frac{\partial \mu_i}{\partial \beta_{*j}} = 0 , \quad j = 0, 1, \ldots k .$$

Since

$$\frac{1}{\mu_i} \frac{\partial \mu_i}{\partial \beta_{*j}} = -x_{ij} \left(\frac{\mu_i}{t_i}\right)^{\delta} , \quad \frac{1}{\mu_i} \frac{\partial \mu_i}{\partial \delta} = \frac{1}{\delta^2} \left\{ \left(\frac{\mu_i}{t_i}\right)^{\delta} - 1 - \delta \log\left(\frac{\mu_i}{t_i}\right) \right\} \qquad (2.4)$$

the maximum likelihood estimates are found as the solution of the $k+2$ equations

$$\sum_{i=1}^{n} x_{ij} (\omega_i - \hat{\mu}_i) \left(\frac{\hat{\mu}_i}{t_i}\right)^{\hat{\delta}} = 0 , \qquad j = 0, 1, \ldots, k \qquad (2.5)$$

$$\sum_{i=1}^{n} (\omega_i - \hat{\mu}_i) \left\{ \left(\frac{\hat{\mu}_i}{t_i}\right)^{\hat{\delta}} - 1 - \hat{\delta} \log\left(\frac{\hat{\mu}_i}{t_i}\right) \right\} = 0 \qquad (2.6)$$

where $\quad \hat{\mu}_i = t_i(1 + \hat{\delta}\underset{\sim}{x}'_{*i}\hat{\underset{\sim}{\beta}}_*)^{-1/\hat{\delta}}$

Routine calculation gives the second order derivatives of the log-likelihood as

$$\frac{\partial^2 \log L}{\partial\beta_{*j}\,\partial\beta_{*j'}} = -\sum_{i=1}^{n} x_{ij}\, x_{ij'}\, \emptyset_i^2 \left\{(\delta+1)\,\mu_i - \delta\omega_i\right\} = -\,a_{jj'}\,, \tag{2.7}$$

$$\frac{\partial^2 \log L}{\partial\beta_{*j}\,\partial\delta} = -\frac{1}{\delta}\sum_{i=1}^{n} x_{ij}\,\emptyset_i\left\{(\omega_i-\mu_i)(\emptyset_i-1) - \mu_i z_i\right\} = -\,b_j \tag{2.8}$$

$$\frac{\partial^2 \log L}{\partial\delta^2} = -\frac{1}{\delta^2}\sum_{i=1}^{n}\left[\mu_i z_i^2 + (\omega_i-\mu_i)\left\{z_i(3-\emptyset_i) - \frac{(\emptyset_i-1)}{\delta}\log\emptyset_i\right\}\right] = -c \tag{2.9}$$

for $j, j' = 0, 1, \ldots, k$, where

$$\emptyset_i = \left(\frac{\mu_i}{t_i}\right)^{\delta}, \qquad z_i = \frac{1}{\delta}\,(\emptyset_i - 1 - \log\emptyset_i) \tag{2.10}$$

Setting

$$\underset{\sim}{A} = ((\hat{a}_{jj'}))\ , \qquad \underset{\sim}{B} = ((\hat{b}_j))\ , \qquad \underset{\sim}{C} = ((\hat{c}))$$

where $\hat{a}_{jj'}$ denotes $a_{jj'}$ evaluated at $\underset{\sim}{\beta}_* = \hat{\underset{\sim}{\beta}}_*$ and $\delta = \hat{\delta}$, etc, the estimated asymptotic covariance matrix of the estimators is

$$\underset{\sim}{V} = \begin{bmatrix} \underset{\sim}{V}_{11} & \underset{\sim}{V}_{12} \\ \underset{\sim}{V}'_{12} & \underset{\sim}{V}_{22} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B}' & \underset{\sim}{C} \end{bmatrix}^{-1}$$

The matrices in the partitioned form are given by

$$\begin{aligned} \underset{\sim}{V}_{11} &= \underset{\sim}{A}^{-1} + \underset{\sim}{A}^{-1}\underset{\sim}{B}(\underset{\sim}{C} - \underset{\sim}{B}'\underset{\sim}{A}^{-1}\underset{\sim}{B})^{-1}\,\underset{\sim}{B}'\underset{\sim}{A}^{-1} \\ \underset{\sim}{V}_{12} &= -\underset{\sim}{A}^{-1}\underset{\sim}{B}(\underset{\sim}{C} - \underset{\sim}{B}'\underset{\sim}{A}^{-1}\underset{\sim}{B})^{-1}, \quad \underset{\sim}{V}_{22} = (\underset{\sim}{C} - \underset{\sim}{B}'\underset{\sim}{A}^{-1}\underset{\sim}{B})^{-1} \end{aligned} \tag{2.11}$$

The calculation of the estimates and statistics used for making inferences about $\delta$ and $\beta_*$ is performed straightforwardly using the statistical package GLIM which has been developed for fitting generalised linear models by maximum likelihood for error distributions in the exponential family. The method is similiar to that described by Aitken and Clayton (1980) for fitting proportional hazard regression models for censored survival data.

In (2.3), the first term represents the kernel of a log-likelihood treating $W_1, W_2, \ldots, W_n$ as independent Poisson variables with $E(W_i) = \mu_i$, and the second term is independent of $\underset{\sim}{\beta}_*$ and $\delta$. From (2.2) we have

$$\{(t_i/\mu_i)^{\delta} - 1\}/\delta = \underset{\sim}{x}'_{*i}\,\underset{\sim}{\beta}_* \tag{2.12}$$

If $\delta$ is known, the value $\hat{\underset{\sim}{\beta}}_*(\delta)$ which maximises the log-likelihood, now denoted by $\log L(\underset{\sim}{\beta}_*(\delta), \delta)$, can be found fitting a Poisson regression model with the user defined link function $g_\delta(\mu_i) = \{(t_i/\mu_i)^{\delta} - 1\}/\delta$. In the usual case when $\delta$ is unknown, the maximum likelihood estimates $\hat{\delta}$ and $\hat{\underset{\sim}{\beta}}_*$ can be found using the following iterative procedure which is based on a general method suggested by Pregibon (1980) for selecting the link function in generalised linear models.

Suppose that $\hat{\delta}_1$ denotes an initial approximation to $\hat{\delta}$. Using the first order Taylor series expansion about the value $\delta = \hat{\delta}_1$ gives the approximation

$$g_\delta(\mu_i) \approx \frac{(t_i/\mu_i)^{\hat{\delta}_1} - 1}{\hat{\delta}_1} + (\delta - \hat{\delta}_1)\left\{\frac{\hat{\delta}_1 (t_i/\mu_i)^{\hat{\delta}_1}\log(t_i/\mu_i) + 1 - (t_i/\mu_i)^{\hat{\delta}_1}}{\hat{\delta}_1^2}\right\} \tag{2.13}$$

An initial fit is made using the model $g_{\hat{\delta}_1}(\mu_i) = \underset{\sim}{x}'_{*i}\underset{\sim}{\beta}_*$ to yield the estimate $\hat{\underset{\sim}{\beta}}{}^{(1)}_*$ and fitted values $\hat{\mu}_i^{(1)}$ from which we form

$$\hat{z}_i^{(1)} = \left\{\hat{\delta}_1\left(t_i/\hat{\mu}_i^{(1)}\right)^{\hat{\delta}_1}\log\left(t_i/\hat{\mu}_i^{(1)}\right) + 1 - \left(t_i/\hat{\mu}_i^{(1)}\right)^{\hat{\delta}_1}\right\}/\hat{\delta}_1^2 \tag{2.14}$$

A fit of the model $g_{\hat{\delta}_1}(\mu_i) = \underset{\sim}{x}'_{*i}\underset{\sim}{\beta}_* + \hat{z}_i^{(1)}(\delta - \hat{\delta}_1)$ then gives the second approximation $\hat{\delta}_2$, where $\hat{z}_i^{(1)}$, is treated as an additional explanatory variable. The procedure is then repeated until satisfactory convergence is obtained.

If a confidence interval for $\delta$ is required, we use the approximate distribution result

$$2 \log L\{\hat{\beta}_*(\hat{\delta}), \hat{\delta}\} - 2 \log L\{\hat{\beta}_*(\delta), \delta\} \xrightarrow{d} \chi_1^2$$

giving

$$P\left[\log L\{\hat{\beta}_*(\delta),\delta\} > \log L(\hat{\beta}_*(\hat{\delta}),\hat{\delta}) - \tfrac{1}{2}\chi_1^2(1-\alpha)\right] \approx 1-\alpha \qquad (2.15)$$

where $\chi_\nu^2(1-\alpha)$ denotes the upper $100\alpha\%$ point of the $\chi_\nu^2$ distribution. Thus if $\delta_L$ and $\delta_U$ are the two values of $\delta$ for which

$$\log L\{\hat{\beta}_*(\delta),\delta\} = \log L\{\hat{\beta}_*(\hat{\delta}),\hat{\delta}\} - \tfrac{1}{2}\chi_1^2(1-\alpha)$$

where $\delta_L < \delta_U$, then the interval $(\delta_L, \delta_U)$ provides an approximate $100(1-\alpha)\%$ confidence interval for $\delta$. The calculation of this interval will require values of $\log L\{\hat{\beta}_*(\delta),\delta\}$ to be determined for a grid of values of $\delta$.

The covariance matrices for $\hat{\beta}_*$ and $\hat{\delta}$ can also be found straightforwardly using (2.11) since the GLIM output will provide the value of $\underset{\sim}{A}^{-1}$, the covariance matrix for $\hat{\beta}_*$ when $\hat{\delta}$ is treated as the true fixed value of $\delta$. Finally, we may wish to test the hypothesis that a specified subset of the explanatory variables have no effect on the expected time to failure. This can be done using a standard likelihood ratio test based on the difference between the maximised log-likelihoods under the full and reduced models, this being the difference between the corresponding deviances in the GLIM output.

## 3. RESIDUAL PLOTS

A graphical procedure based on suitably defined observed residuals provides a useful preliminary assessment of adequacy of fit of the power transformation model. If the model is correct, the random variable $R = T^*(1+\delta \underset{\sim}{x}_*' \underset{\sim}{\beta}_*)^{-1/\delta}$ has the standard exponential distribution. Thus if there is no censoring, we define observed residuals by

$$r_i = t_i(1 + \hat{\delta}\underset{\sim}{x}_{*i}' \hat{\beta}_*)^{-1/\hat{\delta}} \quad , \quad i=1,2,\ldots,n \qquad (3.1)$$

If the model is correct, the $\{r_i\}$ will approximately have the properties expected of a random sample of $n$ observations from the standard exponential distribution. A plot of the ith ordered residual $r_{(i)}$ against the ith smallest exponential score $e_{i,n} = \sum_{n-i+1}^{n} j^{-1}$ should then give an approximate straight line relation with unit slope and zero intercept.

To deal with censoring, we note that

$$E\{R \mid R > t\,(1 + \delta \underset{\sim}{x}_*' \underset{\sim}{\beta}_*)^{-1/\delta}\} = 1 + t(1 + \delta \underset{\sim}{x}_*' \underset{\sim}{\beta}_*)^{-1/\delta} \tag{3.2}$$

Replacing the censored value of $R$ by the maximum likelihood estimate of its expected value, we define modified residuals by

$$r_i^* = \begin{cases} t_i(1 + \hat{\delta} \underset{\sim}{x}_{*i}' \hat{\underset{\sim}{\beta}}_*)^{-1/\hat{\delta}} & \text{if } \omega_i = 1 \\ 1 + t_i(1 + \hat{\delta} \underset{\sim}{x}_{*i}' \hat{\underset{\sim}{\beta}}_*)^{-1/\hat{\delta}} & \text{if } \omega_i = 0 \end{cases} \tag{3.3}$$

A plot of $r_{(i)}^*$ against $e_{i,n}$ should again give an approproximate straight line relation if the assumed power transformation model is correct.

## 4. AN ILLUSTRATIVE EXAMPLE

To illustrate the use of the power transformation model we use the data in table 1 which were taken from Prentice (1973). The data are for groups of advanced lung cancer patients, the groups being defined by two factors, type of chemotherapeutic agent at two levels standard and test, and tumour cell type at four levels, squamous, small, adeno and large. In the original study, four explanatory variables were measured and the analysis based on model (1.7) indicated that only one variable, general medical status on a scale 10,20,...,90 had a real effect. We have only included this variable in our analysis. The survival times are in days and right censored observations are shown with an asterisk.

The GLIM fitting procedure described in section 2 was used to fit the model.

$$\mu_x(j) = [1 + \delta\{\beta_{*0}(j) + x\beta_{*1}(j)\}]^{1/\delta}, \quad j=1,2,\ldots,8 \tag{4.1}$$

If the model is correct, the $\{r_i\}$ will approximately have the properties expected of a random sample of $n$ observations from the standard exponential distribution. A plot of the ith ordered residual $r_{(i)}$ against the ith smallest exponential score $e_{i,n} = \sum_{n-i+1}^{n} j^{-1}$ should then give an approximate straight line relation with unit slope and zero intercept.

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## TABLE 1

Data for lung cancer patients: t = days of survival, x = general medical status, $G_1$ = standard squamous, $G_2$ = standard small, $G_3$ = standard adeno, $G_4$ = standard large, $G_5$ = test, squamous, $G_6$ = test, small, $G_7$ = test, adeno, $G_8$ = test, large.

| $G_1$ | | $G_2$ | | $G_3$ | | $G_4$ | | $G_5$ | | $G_6$ | | $G_7$ | | $G_8$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| t | x | t | x | t | x | t | x | t | x | t | x | t | x | t | x |
| 72 | 60 | 30 | 60 | 8 | 20 | 177 | 50 | 999 | 90 | 25 | 30 | 24 | 40 | 52 | 60 |
| 411 | 70 | 384 | 60 | 92 | 70 | 162 | 80 | 112 | 80 | 103* | 70 | 18 | 40 | 164 | 70 |
| 228 | 60 | 4 | 40 | 35 | 40 | 216 | 50 | 87* | 80 | 21 | 20 | 83* | 90 | 19 | 30 |
| 126 | 60 | 54 | 80 | 117 | 80 | 553 | 70 | 231* | 50 | 13 | 30 | 31 | 80 | 53 | 60 |
| 118 | 70 | 13 | 60 | 132 | 80 | 278 | 60 | 242 | 50 | 87 | 60 | 51 | 60 | 15 | 30 |
| 10 | 20 | 123* | 40 | 12 | 50 | 12 | 40 | 991 | 70 | 2 | 40 | 90 | 60 | 43 | 60 |
| 82 | 40 | 97* | 60 | 162 | 80 | 260 | 80 | 111 | 70 | 20 | 30 | 52 | 60 | 340 | 80 |
| 110 | 80 | 153 | 60 | 3 | 30 | 200 | 80 | 1 | 20 | 7 | 20 | 73 | 60 | 133 | 70 |
| 314 | 50 | 59 | 30 | 95 | 80 | 156 | 70 | 587 | 60 | 24 | 60 | 8 | 50 | 111 | 60 |
| 100* | 70 | 117 | 80 | | | 182* | 90 | 389 | 90 | 99 | 70 | 36 | 70 | 231 | 70 |
| 42 | 60 | 16 | 30 | | | 143 | 90 | 33 | 30 | 8 | 80 | 48 | 10 | 378 | 80 |
| 8 | 40 | 151 | 50 | | | 105 | 80 | 25 | 20 | 99 | 80 | 7 | 40 | 49 | 30 |
| 144 | 30 | 22 | 60 | | | 103 | 80 | 357 | 70 | 61 | 70 | 140 | 70 | | |
| 25* | 80 | 56 | 80 | | | 250 | 70 | 467 | 90 | 25 | 70 | 186 | 90 | | |
| 11 | 70 | 21 | 40 | | | 100 | 60 | 201 | 80 | 95 | 70 | 84 | 80 | | |
| | | 18 | 20 | | | | | 1 | 50 | 80 | 50 | 19 | 50 | | |
| | | 139 | 80 | | | | | 30 | 70 | 51 | 30 | 45 | 40 | | |
| | | 20 | 30 | | | | | 44 | 60 | 29 | 40 | 80 | 40 | | |
| | | 31 | 70 | | | | | 283 | 90 | | | | | | |
| | | 52 | 70 | | | | | 15 | 50 | | | | | | |
| | | 287 | 60 | | | | | | | | | | | | |
| | | 18 | 30 | | | | | | | | | | | | |
| | | 51 | 60 | | | | | | | | | | | | |
| | | 122 | 80 | | | | | | | | | | | | |
| | | 27 | 60 | | | | | | | | | | | | |
| | | 54 | 70 | | | | | | | | | | | | |
| | | 7 | 50 | | | | | | | | | | | | |
| | | 63 | 50 | | | | | | | | | | | | |
| | | 392 | 40 | | | | | | | | | | | | |
| | | 10 | 40 | | | | | | | | | | | | |

where $\mu_x(j)$ denotes the expected survival time for a patient in group j having medical status value x. The use of the initialisation value $\hat{\delta}_1 = 0$ led to the sequence $\hat{\delta}_2 = 0.359$, $\hat{\delta}_3 = 0.432$, $\hat{\delta}_4 = 0.429$ of approximations to $\hat{\delta}$ and $\hat{\delta}$ was therefore taken as 0.43 correct to two decimal places. In table 2, values of $\hat{\mu}_x(j)$ are given for the eight groups and the nine medical status levels used for classifying the patients. The maximum likelihood estimates of $\beta_{*0}(j)$ and $\beta_{*1}(j)$ are also shown together with their estimated standard errors. The results show that at the higher levels of x, the expected survival times are

highest for group G5, and that the differences among the groups G3, G6 and G7 are relatively small. These results support the general finding made by Prentice and it should be noted that for these data, the use of the power transformation model leads to only a moderate improvement in fit compared with the model given by (1.7). The value of $2\log L\{(\hat{\beta}_*(0.43), 0.43\} - 2\log L\{\hat{\beta}_*(0),0\}$ is 1.41 which when referred to the $\chi^2_1$ distribution is just significant at the 25% level. It is of course instructive to use the general power transformation model to assess the adequacy of fit of the commonly used model given by (1.7).

TABLE 2

Maximum likelihood estimates of regression coefficients and of expected survival times.

| x | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $G_6$ | $G_7$ | $G_8$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 27.4 | 43.0 | 1.4 | 49.2 | 1.9 | 9.3 | 23.8 | 2.5 |
| 20 | 43.6 | 52.1 | 5.6 | 67.4 | 13.8 | 15.5 | 30.1 | 10.3 |
| 30 | 64.1 | 62.2 | 13.0 | 88.8 | 39.4 | 23.6 | 37.1 | 24.8 |
| 40 | 89.2 | 73.4 | 24.2 | 113.7 | 81.2 | 33.7 | 45.1 | 46.8 |
| 50 | 119.2 | 85.6 | 39.5 | 142.1 | 140.7 | 46.0 | 53.9 | 77.1 |
| 60 | 154.2 | 98.9 | 59.2 | 174.3 | 219.4 | 60.4 | 63.6 | 116.2 |
| 70 | 194.5 | 113.3 | 83.5 | 210.2 | 318.6 | 77.1 | 74.3 | 164.7 |
| 80 | 240.1 | 128.8 | 112.7 | 250.0 | 439.4 | 96.1 | 85.9 | 223.0 |
| 90 | 291.4 | 145.5 | 146.9 | 293.7 | 582.7 | 117.6 | 98.5 | 291.7 |
| $\hat{\beta}_{*0}(j)$ | 5.21 | 8.38 | -1.74 | 8.30 | -3.36 | 2.23 | 5.81 | -1.80 |
| $SE\{\hat{\beta}_{*0}(j)\}$ | 6.76 | 4.83 | 2.61 | 13.75 | 3.23 | 2.68 | 2.86 | 4.48 |
| $\hat{\beta}_{*1}(j)$ | 0.213 | 0.101 | 0.214 | 0.179 | 0.411 | 0.150 | 0.096 | 0.291 |
| $SE\{\hat{\beta}_{*1}(j)\}$ | 0.140 | 0.090 | 0.072 | 0.204 | 0.899 | 0.067 | 0.055 | 0.110 |

## 5. EXTENSIONS

The regression model given by (1.8) for the mean or equivalently the reciprocal of the hazard function of the exponential distribution is more flexible than other models that have been previously used. It is therefore likely that except for very large data sets, the model will provide a satisfactory fit if the underlying distribution is exponential, even if the model's link function is not correct.

For distributions other than the exponential the power transformation link function may be used to model the dependence of the hazard function

on explanatory variables. Thus if $h(t; \underset{\sim}{x})$ denotes the hazard function for time to failure given $\underset{\sim}{x}$, we may use the proportional hazards regression model

$$h(t;\underset{\sim}{x}) = h_o(t;\underset{\sim}{\theta})(1+\delta\underset{\sim}{x}'\underset{\sim}{\beta})^{1/\delta} \tag{5.1}$$

where $h_o(t;\underset{\sim}{\theta})$ represents the hazard function for an individual with $\underset{\sim}{x} = \underset{\sim}{0}$ which is assumed to depend on a vector $\underset{\sim}{\theta}$ of unknown parameters.

A straightforward argument shows that the log-likelihood under the model given by (5.1) is

$$\log L(\underset{\sim}{\beta},\underset{\sim}{\theta},\delta) = \sum_{i=1}^{n}(\omega_i \log \mu_i - \mu_i) + \sum_{i=1}^{n}\omega_i \log\{h_o(t_i;\underset{\sim}{\theta})/H_o(t_i;\underset{\sim}{\theta})\} \tag{5.2}$$

where

$$H_o(t;\underset{\sim}{\theta}) = \int_0^t h_o(u;\underset{\sim}{\theta})\,du \tag{5.3}$$

is the cumulative hazard function and

$$\mu_i = (1+\delta\underset{\sim}{x}_i'\underset{\sim}{\beta})^{1/\delta} H_o(t;\underset{\sim}{\theta}), \qquad i=1,\ldots,n \tag{5.4}$$

Suppose now that the underlying distribution depends on two parameters, $\lambda$ and $\theta$ say, and that

$$H_o(t;\lambda,\theta) = \lambda\eta(t;\theta) \tag{5.5}$$

where $\lambda > 0$ and $\eta(t;\theta) \geq 0 \; \forall t > 0$. Examples of distributions having this property are the Weibull with $H_o(t;\lambda,\theta) = \lambda t^{\theta}$ and the Gompertz with $H_o(t;\lambda,\theta) = (\lambda/\theta)\exp(\theta t)$, $t > 0$. If (5.5) holds, we may write

$$\mu_i = \eta(t_i;\theta)(1+\delta\underset{\sim}{x}_{*i}'\underset{\sim}{\beta}_*)^{1/\delta} \tag{5.6}$$

where

$$\beta_{*0} = \frac{\lambda^{\delta}-1}{\delta}, \qquad \beta_{*j} = \lambda^{\delta}\beta_j, \qquad j = 1,\ldots,k \tag{5.7}$$

For $\theta$ known, the ML estimate $\underset{\sim}{\beta}_*(\theta)$ can be found treating the $\{\omega_i\}$ as observations on independent Poisson regression variables, the link function being

$$g(\mu_i) = \left[\left\{\frac{\mu_i}{\eta(t_i;\theta)}\right\}^{\delta} - 1\right] \Big/ \delta \tag{5.8}$$

If $\theta$ is unknown, a search over a grid of values of $\theta$ would be needed to find the global maximum of the log-likelihood.

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